Separability and Entanglement of Identical Bosonic Systems

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Abstract We investigate the separability of arbitrary n-dimensional multipartite identical bosonic systems. An explicit relation between the dimension and the separability is presented. In particular, for $n = 3$, it is shown that the property of PPT (positive partial transpose) and the separability are equivalent for tripartite systems.

Key words: Separability, Quantum entanglement, PPT state
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Quantum entanglement plays essential roles in quantum information processing and quantum computation. The entangled states provide key resources for a vast variety of novel phenomena such as quantum cryptography, quantum teleportation, super dense coding, etc [1]. An important problem in the theory of quantum entanglement is the separability. One of the famous separability criterion was given by Peres [2]. It says that all separable states necessarily have a positive partial transpose (PPT), which is further shown to be also sufficient for states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ [3, 4], where $\mathbb{C}^n$ denotes the n-dimensional complex space. There have been many results on the separability and entanglements of mixed states, see e.g., [5, 6, 7, 8, 9]. In particular, it is shown that every quantum states ρ supported on $\mathbb{C}^M \otimes \mathbb{C}^N$, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$ and $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^N$ with positive partial transposes and rank $r(\rho) \leq N$ are separable and have a canonical form [5, 6, 7].

Although the entanglement is extensively studied for distinguishable particle systems, the entanglement of identical particle systems has been less investigated. In fact in certain systems such as quantum dots [10], Bose-Einstein condensates [11] and parametric down conversion [12], the entanglement should be treated as the one of identical particle systems. Schliemann et al [10, 13] have discussed the entanglement in two-fermion systems. They found that the entanglement in two-fermion systems is analogous to that in a two-distinguishable particle system. The results for two-boson systems are quite different. Li *et al.* [14] and Paskauskas and You [15] have studied this problem of two-boson systems. For multipartite bosonic systems, there are very few discussions. Recently, the author in [16] obtained the canonical form for pure states of three identical bosons, and classified the

entanglement correlation into two types, the analogous GHZ and the W-types. In [17], it has been shown that rank n and rank $\frac{n(n+1)}{2}-2$ PPT bosonic mixed states in the symmetrized tensor product space $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n)$ are separable, and all three-qubit ($n=2$) bosonic PPT states are separable as well. For bosonic mixed state ρ in $k-$qubit system, $k \geq 4$, ρ is PPT implies that ρ is separable, except for the case of maximal rank.

In this letter, we investigate the separability of multi-partite identical bosonic systems with arbitrary dimension n. Let $\mathcal{H} = \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ denote the symmetrized tensor product space of k n-dimensional spaces associated with Alice, Bob, Charle, etc. The dimension of the space $\mathcal{H}$ is given by [18],

$$I_n^k = \frac{(n+k-1)!}{k!(n-1)!} = C_{n+k-1}^k. \tag{1}$$

We first consider the case of $k=3$.

[**Theorem 1**] Let ρ be a bosonic mixed state in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)$, with a positive partial transpose with respect to Alice. If the rank of ρ, $r(\rho) \leq n^2$, then ρ is separable.

Proof. We first prove the case of $n=3$. Suppose that the state ρ is a PPT state with respect to Alice and has a rank 9. We can treat it as a bipartite PPT state in a 3×9 dimensional space of Alice-(Bob,Charlie). From the Theorem 1 in [5] (also Theorem 1 in [6]), such a state of rank 9 is necessarily separable and can be represented as $\rho = \sum_{i=1}^9 p_i |e_i, \Psi_i\rangle\langle e_i, \Psi_i|$, where the vectors $|\Psi_i\rangle$ are generally entangled pure states associated with the spaces of Bob and Charlie. As $|\Psi_i\rangle$ are mutually orthogonal, they belong to the range of the reduced density matrix (partial trace with respect to the space associated with Alice) $\mathrm{Tr}_A \rho$, and hence $|\Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. Moreover $|e_i, \Psi_i\rangle$ belong to the range of ρ. Therefore $|e_i, \Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. According to Schmidt decomposition we can write $|\Psi_i\rangle = a_i|00\rangle + b_i|11\rangle + c_i|22\rangle$ for some $a_i, b_i, c_i \in \mathbb{C}$, where $|0\rangle, |1\rangle, |2\rangle$, are the Schmidt basic vectors in $\mathbb{C}^3$. The only possible forms of $|e_i, \Psi_i\rangle$ satisfying the above conditions are $|000\rangle$, $|111\rangle$ or $|222\rangle$. Therefore ρ is separable.

When the rank of ρ is strictly less then 9, ρ can be embedded into a smaller space. For instance, if $r(\rho) = 8$, ρ is supported on spaces 2×8 or 3×8. ρ is then separable in the partition Alice-(Bob,Charlie) and can be again written as $\rho = \sum_{i=1}^8 p_i |e_i, \Psi_i\rangle\langle e_i, \Psi_i|$. By using the same procedure as above, we can prove that $|e_i, \Psi_i\rangle$ is fully separable, and hence ρ is separable. The general n-dimensional case can be proved similarly. $\square$

[**Remark**] From the theorem we see that a bosonic mixed state ρ in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ with a positive partial transpose is separable if $r(\rho) \leq 9$. As the dimension of the space of $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ is 10, Theorem 1 says that almost all the PPT bosonic mixed states in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ are separable, except for the case $r(\rho) = 10$. Hence the rank of a bound entangled state in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ has to be 10.

When $n=4$, we have $I_4^3 = 20$. As ρ is separable if $r(\rho) \leq 16$, all bound entangled states ρ in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)$ satisfy $17 \leq r(\rho) \leq 20$.

[Theorem 2] Let ρ be a PPT bosonic mixed state in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ with k subsystems ($k \geq 4$). If $r(\rho) \leq I_n^{k-1}$, then ρ is separable.

Proof. We prove the case of $n = 3$ (the other cases can be proved similarly). Assume that ρ is PPT, say with respect to the space associated with Alice, with rank $I_3^{k-1} = \frac{k(k+1)}{2}$.

If we consider ρ as a bipartite state in the partition Alice - the rest, ρ is supported on $\mathbb{C}^3 \otimes \mathcal{S}((\mathbb{C}^3)^{\otimes k-1})$. From [5] ρ is separable with respect to this partition and has a form, $\rho = \sum_{i=1}^{\frac{k(k+1)}{2}} p_i |e_i, \Psi_i\rangle\langle e_i, \Psi_i|$, where $|e_i\rangle$ (resp. $|\Psi_i\rangle$) are vectors on the spaces associated to Alice (resp. the rest).

We prove result by induction. We illustrate the procedure by proving the case of $k = 4$. As $|\Psi_i\rangle$ belong to the range of the reduced density matrix $\mathrm{Tr}_A\rho$, they must belong to $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. Since ρ is PPT $|\Psi_i\rangle\langle\Psi_i|$ is a PPT state in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. However the rank $r(|\Psi_i\rangle\langle\Psi_i|) = 1$, from Theorem 1, $|\Psi_i\rangle$ is separable, and can written as $|\Psi\rangle = |f_i, f_i, f_i\rangle$ for some vectors $|f_i\rangle$ in $\mathbb{C}^3$. While the vectors $|e_i, \Psi_i\rangle$ belong to the range of ρ and hence $|e_i, \Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. Therefore the only possible forms of $|e_i, \Psi_i\rangle$ are $|f_i, f_i, f_i, f_i\rangle$. Therefore ρ is separable. $\square$

We have presented some separability criteria for multipartite bosonic mixed states. For tripartite PPT states, all bound entangled states have necessarily rank greater than n^2. For general multipartite PPT bosonic states with k subsystems ($k \geq 4$), if $r(\rho) \leq I_n^{k-1}$, ρ is separable. The results can be used to construct possible bound entangled states of identical bonsonic systems. For instance, if $k = 4$, $n = 3$, we have $I_3^4 = 15$. The rank of a bound entangled state has to be between $I_3^3 = 10$ and 15.

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